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On the damped harmonic oscillator in the de Broglie–Bohm hidden-variable theory

M A Vandyck

Physics Department, University College, Cork, Ireland, and Physics Department, Cork Regional Technical College, Bishopstown, Co. Cork, Ireland

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Abstract. The problem of the harmonic oscillator damped proportionally to the velocity is investigated in the de Broglie–Bohm hidden-variable theory. (This system is known to be *classically equivalent to a particular type of undamped oscillator with variable mass.*) The de Broglie–Bohm equation of motion is solved explicitly and compared with its classical equivalent. The influence of the 'quantum potential' is discussed. In contrast to some other studies of the de Broglie–Bohm theory, the results obtained here are derived without approximation.

1. Introduction

Among the formalisms that attempt to re-interpret quantum mechanics so as to give a meaning to the notion of the 'trajectory' of an individual particle, two have so far gained a certain acceptance, namely the de Broglie–Bohm causal theory [1–5] and Nelson's stochastic mechanics [6]. Hereafter, we shall exclusively consider the former.

The de Broglie–Bohm theory has been studied from various points of view, and one aspect that has been particularly emphasized is the concept of 'quantum potential'. This tool enables one to consider the motion of a system as following essentially the laws of classical dynamics, but under the influence of a potential composed of the classical potential and the so-called 'quantum potential', a nonlinear contribution coming from the quantum-mechanical wavefunction. This approach has been employed to analyse several problems such as quantum interference [7], the Aharonov–Bohm effect [8], statistical properties [9, 10], the process of measurement in quantum mechanics [11], etc. The de Broglie–Bohm equation of motion has also been solved, mostly numerically, in some simple cases such as time-dependent scattering from square barriers and square potential wells [12, 13]. Explicit exact solutions of the equation of motion are, in general, difficult to obtain for non-trivial systems because of the nonlinearity of the quantum potential.

One particular system which is important for physical reasons is the harmonic oscillator. We shall present hereafter an analysis of the *damped* harmonic oscillator according to the de Broglie–Bohm theory. (The damping force is taken proportional to the velocity.) This system has the property that, in spite of the nonlinearity of the quantum potential, it is possible to solve exactly the de Broglie–Bohm equation of motion.

Moreover, the classical *damped* oscillator possesses, as is well known, two types of qualitatively different trajectories, depending on the relative values of the parameters, the damping constant and the natural angular frequency of oscillation: the trajectory of the underdamped system exhibits oscillations, whereas neither the overdamped nor the

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critically damped system oscillates at all. The question may then be asked whether the same relationship between the strength of the damping and the type of trajectory holds for the de Broglie–Bohm oscillator. We shall see that such is not the case and that the classical categories 'underdamped' and 'critically damped' mix in the de Broglie–Bohm theory. In this study, it is again the quantum potential that will shed light onto the physical reasons for the behaviour of the system. These considerations will be presented in section 3.

It is important to note that the *damped* harmonic oscillator with constant mass has the same classical equation of motion [14–16] as a particular type of *undamped* oscillator with variable mass. Our classical treatment will thus apply simultaneously to both problems. (This equivalence, however, does not seem to be maintained in the quantum-mechanical context [16, 17].) For the sake of brevity, we shall always refer to the problem as the 'damped harmonic oscillator'.

According to de Broglie–Bohm theory, before being able to express the equation of motion of a system, one must first solve the corresponding problem in ordinary quantum mechanics. The question of quantizing a system with friction has been investigated by many authors, for instance [15–29]. We shall therefore provide, in section 2, a brief overview of the damped harmonic oscillator in Schrödinger's theory, emphasizing the aspects of the question that will be of importance for the de Broglie–Bohm theory.

2. Damped harmonic oscillator in classical mechanics and Schrödinger's theory

The purpose of this section is two-fold. In the first instance, we shall briefly recall the Lagrangian and Hamiltonian formulations of the classical damped harmonic oscillator, and secondly we shall treat the problem of the damped oscillator according to Schrödinger's theory. We emphasize that we shall deal exclusively with the *underdamped* oscillator, the other cases being similar. In section 3, we shall turn to the treatment of the system according to the de Broglie–Bohm theory.

Let a point P of mass m move under the influence of an ideal spring having a stiffness constant $k \equiv m\omega_0^2$ and of a damping force F_d given by $F_d = -2m\gamma\dot{x}$, where ω_0, γ, and x denote respectively the angular frequency of the undamped oscillator, the damping constant, and the displacement of P from its equilibrium position. The equation of motion of this system reads

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0. \tag{2.1}$$

It has long been known [26, 30–33] that (2.1) derives from at least two non-trivially distinct Lagrangian functions L and L_H given by

$$2L(x, \dot{x}, t) = m\mathrm{e}^{2\gamma t}(\dot{x}^2 - \omega_0^2 x^2) \tag{2.2}$$

$$\mathcal{E}^{-1} L_H(x, \dot{x}, t) = \frac{\dot{x} + \gamma x}{x\omega} \tan^{-1}\left(\frac{\dot{x} + \gamma x}{x\omega}\right) - \frac{1}{2}\ln\left[(\dot{x}^2 + 2\gamma x\dot{x} + \omega_0^2 x^2)m/\mathcal{E}\right] \tag{2.3}$$

in which $\mathcal{E}$ denotes an arbitrary constant, with dimensions of energy, and ω is the angular frequency of the damped oscillator.

Because of the fact that L and L_H yield the same equation of motion, no loss of generality arises from choosing (2.2), as opposed to (2.3), to describe the harmonic oscillator with friction, and we shall make this choice from now on. The corresponding Hamiltonian function reads

$$2mH(x, p_x, t) = \mathrm{e}^{-2\gamma t} p_x^2 + m^2\omega_0^2 \mathrm{e}^{2\gamma t} x^2 \qquad p_x \equiv \partial L/\partial\dot{x}. \tag{2.4}$$

The question of quantizing a dissipative system has been investigated by many authors, and there exist in the literature several review articles, for instance [16, 17, 26, 28, 29], containing comprehensive lists of references. Therefore it is sufficient to present here only a very brief summary of some aspects of the problem.

The most straightforward method consists in quantizing canonically [34] the classical Hamiltonian function H and in solving the resulting Schrödinger equation. At this stage, the question arises as to which of the Hamiltonian functions that give rise to the correct equation of motion (2.1) must be employed [26, 35]. It is indeed clear that the Lagrangian functions (2.2) and (2.3) will lead to different Hamiltonian functions and thus to inequivalent quantizations *via* Schrödinger's equation.

Various points of view have been expressed on how to select the 'right' Hamiltonian function [15, 28], and other quantization schemes have been proposed to circumvent the problem of the choice of the Lagrangian [16, 17, 25, 28, 29, 36]. Without entering into any detailed discussions of this question, we simply state that we adopt here the approach of quantizing canonically the Hamiltonian function (2.4) and solving Schrödinger's equation. The final result of this procedure, the wave function $\Psi(x, t)$ normalized to unity, then reads [17–23, 29]

$$\Psi_n(x, t) = M_n(x, t)P_n(x, t) \tag{2.5}$$

$$M_n \equiv \frac{A_n}{\sqrt{\sigma}} H_n\left(\frac{x}{\sigma}\right) \exp\left(-\frac{x^2}{2\sigma^2}\right) \tag{2.6}$$

$$P_n \equiv \exp\left[-\mathrm{i}\left(\frac{\epsilon_n}{\hbar}t + \frac{\gamma x^2}{2\omega\sigma^2}\right)\right] \tag{2.7}$$

where H_n denotes the nth Hermite polynomial and the quantities A_n, σ, ϵ_n are related to the parameters of the problem by

$$A_n \equiv \left(\sqrt{\pi}2^n n!\right)^{-1/2} \qquad \sigma \equiv (\hbar/m\omega)^{1/2}\mathrm{e}^{-\gamma t} \qquad \epsilon_n \equiv (n + \tfrac{1}{2})\hbar\omega. \tag{2.8}$$

It should be noted that ϵ_n in (2.8) should not be interpreted as the energy of the nth 'energy level' since, given that the system is dissipative, there can be no stationary state. For convenience we shall, however, refer to Ψ_n as the nth 'state' of the oscillator. This terminology will not create any confusion.

It is important to emphasize that the *mathematical* correctness and self-consistency of the various quantization schemes is not questioned; only the *physical* interpretation of the wavefunctions thus produced requires care. Therefore, the point of view adopted here is the following: we need a wavefunction for the oscillator to be able to express the de Broglie–Bohm equation of motion in the following section. It is mathematically self-consistent to use the wavefunction (2.5)–(2.8), and this wavefunction obviously describes the undamped oscillator [37] in the special case $\gamma = 0$. It is thus meaningful to select it as 'guiding wave' for the de Broglie–Bohm theory, bearing in mind that the *physical* validity of the results rests, ultimately, on the *physical* interpretation given to the quantum-mechanical wavefunction. If one wishes, the treatment of the de Broglie–Bohm equation of motion below may always be repeated for different 'guiding waves' obtained by other quantization methods.

3. de Broglie–Bohm damped harmonic oscillator

According to the de Broglie–Bohm theory [1–5], a system possesses a well defined trajectory as it does in classical mechanics. In order to obtain the equation of motion of the system, one must first solve the corresponding Schrödinger equation for the wavefunction $\Psi(q,t)$ and construct a function S, generalizing Hamilton's principal function, by the prescription

$$e^{iS/\hbar} = \Psi/\left(\overline{\Psi}\Psi\right)^{1/2}. \tag{3.1}$$

The equation of motion is then assumed to be the classical relationship between the momentum $p \equiv \partial L/\partial\dot{q}$ of the system and Hamilton's principal function $S(q,t)$, namely

$$p = \partial S(q,t)/\partial q. \tag{3.2}$$

This prescription constitutes a physically reasonable base for a theory of mechanics since it implies [1–4] that S defined by (3.1) fulfils the Hamilton–Jacobi equation in the limit $\hbar \longrightarrow 0$. More precisely, for the Lagrangian function of interest here, namely

$$L(x,\dot{x},t) = \tfrac{1}{2}me^{2\gamma t}\dot{x}^2 - V(x,t) \tag{3.3}$$

which contains (2.2) as the special case

$$V(x,t) = \tfrac{1}{2}m\omega_0^2 e^{2\gamma t}x^2 \tag{3.4}$$

one may construct the Hamiltonian function and Schrödinger's equation. If the wavefunction $\Psi(x,t)$ is decomposed into its modulus $M(x,t)$ and its phase as in (3.1), the real and imaginary parts of Schrödinger's equation, respectively, then read

$$\frac{e^{-2\gamma t}}{2m}(S_{,x})^2 + \left(V - \frac{\hbar^2}{2m}e^{-2\gamma t}\frac{M_{,xx}}{M}\right) + S_{,t} = 0 \tag{3.5}$$

$$\left(M^2\frac{S_x}{m}e^{-2\gamma t}\right)_{,x} + \left(M^2\right)_{,t} = 0. \tag{3.6}$$

In the limit $\hbar \longrightarrow 0$, (3.5) becomes the classical Hamilton–Jacobi equation, so that (3.2) yields the classical equation of motion. (It should be noted that (3.5) and (3.6) are slightly different from the usual de Broglie–Bohm equations [3, 4]. This is due to the fact that the systems usually studied are assumed frictionless, so that the constant γ of (3.3) is normally considered as vanishing. As one can see, there is no complication in treating the more general (3.3).)

On the other hand, when the limit $\hbar \longrightarrow 0$ is not taken, (3.5) shows that the system seems to move according to the classical equation of motion, but under the influence of a potential composed of the classical V and the 'quantum' potential Q, nonlinear in the wavefunction, defined by

$$Q(x,t) \equiv \frac{-\hbar^2}{2m}e^{-2\gamma t}\frac{M_{,xx}}{M}, \tag{3.7}$$

which is a given function of x and t after substitution of the function M obtained by solving Schrödinger's equation. Explicitly, if one returns to the Lagrangian formalism, the equation of motion for the Lagrangian function (3.3) becomes, in terms of V and Q

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{x}} + \frac{\partial}{\partial x}(V+Q) = 0, \tag{3.8}$$

which is equivalent to

$$m\ddot{x} + 2m\gamma\dot{x} + \frac{\partial}{\partial x}\left[(V + Q)e^{-2\gamma t}\right] = 0. \tag{3.9}$$

It should be emphasized, however, that this terminology is somewhat misleading since, if one were to study the classical motion of a system under the total potential $V + Q$, one would solve *only* the Hamilton–Jacobi equation (3.5) for S and proceed from there. This would indeed lead to an equation of motion equivalent to (3.9). On the other hand, the de Broglie–Bohm theory requires S to be the solution of *not only* the Hamilton–Jacobi equation (3.5) but *also* of (3.6). Consequently, S is *less general* in the de Broglie–Bohm theory, having to satisfy two equations, than it would be in the classical problem for the total potential $V + Q$. Thus, in general, the de Broglie–Bohm trajectory, solution of (3.2), is a *special case* of the classical trajectory (3.9) arising from the potential $V + Q$. As we shall see, this is precisely what happens for the damped harmonic oscillator.

An important consequence of (3.1) is that only the phase of the wavefunction Ψ contributes to S. Moreover, whenever the phase of the wavefunction is independent of the spatial coordinate q, so is the corresponding S, and the equation of motion (3.2) reduces to $p = 0$, predicting that the system is 'at rest'. Such is the case for the usual frictionless harmonic oscillator [37]: energy is conserved, which implies that Schrödinger's equation admits stationary states of the form $\Psi_n(x, t) = \psi_n(x)\exp(-iE_nt/\hbar)$. Furthermore, the time-independent wavefunctions $\psi_n(x)$ happen to be purely real [37], and therefore the phase of $\Psi_n(x, t)$ is independent of the position, with the consequence just mentioned. On the other hand, the wavefunction Ψ_n of the damped oscillator (2.5)–(2.8) does possess a non-vanishing position-dependent phase thanks to the friction, so that the de Broglie–Bohm trajectory is non-trivial, as we shall see.

It is a simple matter to substitute (2.5)–(2.7) into (3.1), (3.2), calculate the derivative, use the definition (2.8) of σ, and finally obtain the equation of motion in the form

$$me^{2\gamma t}\dot{x} = p = -m\gamma e^{2\gamma t}x. \tag{3.10}$$

The solution of (3.10) may be written

$$x(t) = x_0e^{-\gamma t} \tag{3.11}$$

where x_0 is the initial position of the oscillator.

The trajectory possesses the property that it is independent of both the quantum number n appearing in the wavefunction (2.5)–(2.7) and of the mass m of the oscillator, although it does depend on the friction constant γ. It is also monotonically approaching the origin $x = 0$ as the time increases. Such a behaviour is very different from that of an underdamped harmonic oscillator according to classical mechanics, but all these features can be explained by investigating the quantum potential Q.

If the expression (2.6) for M_n is inserted into the definition (3.7) of the quantum potential, and the differential equation satisfied by the Hermite polynomials is taken into account, as well as the definition (2.8) of σ, it is found that the quantum potential reads

$$Q(x, t) = -\tfrac{1}{2}me^{2\gamma t}\omega^2x^2 + (n + \tfrac{1}{2})\hbar\omega. \tag{3.12}$$

As one can see, the quantum potential of this problem is obtained without approximation. This contrasts with what happens for other physical systems treated according to the

de Broglie–Bohm theory. When this explicit quantum potential is added to the classical potential V of (3.4), a non-trivial cancellation takes place between the terms involving ω_0^2 and ω^2, yielding as total potential

$$V + Q = \tfrac{1}{2}me^{2\gamma t}\gamma^2 x^2 + (n + \tfrac{1}{2})\hbar\omega. \tag{3.13}$$

Let us now consider a classical system moving under the influence of this total potential, namely according to the equation of motion (3.9). The latter becomes

$$\ddot{x} + 2\gamma\dot{x} + \gamma^2 x = 0 \tag{3.14}$$

which is the equation of a classical harmonic oscillator *critically damped* since the natural angular frequency ω_0 is equal to the damping constant γ. The fact that the damping is critical could already have been deduced from the form of the potential (3.13), where γ^2 appears in place of what would be ω_0^2 for an underdamped oscillator. It is also clear from (3.13) that the trajectory must be independent of the quantum number n because of the fact that the n-term contributes additively to the total potential and is independent of x. Finally, it remains to be noted that the de Broglie–Bohm equation of motion (3.10) and its solution (3.11) are special cases of the trajectory (3.14) of the classical system moving under the influence of $V+Q$. We have thus proved that the de Broglie–Bohm *underdamped* harmonic oscillator behaves like a particular classical *critically damped* oscillator. The connection is established through the quantum potential which makes the interpretation transparent, and the link that we found between the de Broglie–Bohm trajectory and the classical trajectory enables us to explain the qualitative features of the motion, on which we shall briefly comment.

In the first instance, the reason why the de Broglie–Bohm underdamped oscillator approaches the origin without oscillating is obvious: mathematically, the underdamped de Broglie–Bohm oscillator is a critically damped classical oscillator, and the latter does not oscillate. From the physical point of view, an analogy with the frictionless de Broglie–Bohm oscillator is enlightening. Indeed, as emphasized above in the second paragraph after (3.9), the frictionless harmonic oscillator guided by a quantum stationary state Ψ_n is 'at rest' according to the de Broglie–Bohm theory. Thus, in this case, although the classical trajectory does oscillate (with constant amplitude), the de Broglie–Bohm trajectory is not oscillatory at all. The situation would be different if one were to use as a guiding wave a linear superposition of the wavefunctions Ψ_m and Ψ_n of two distinct energy levels m and n, $m \neq n$. (Unfortunately, even in the frictionless case, the equation of motion (3.2) cannot be solved analytically for such a superposition.)

Seen in this light, the behaviour (3.11) of the harmonic oscillator with friction according to the de Broglie–Bohm theory is not unexpected: we used as a guiding wave not a superposition of 'states' but one single 'state' Ψ_n, n fixed. Given that the friction introduces into the guiding wave (2.5)–(2.7) a position-dependent phase, the de Broglie–Bohm system is no longer at rest, and the trajectory is non-trivial. Oscillations are absent, as they are in the frictionless case for a fixed n. One may also note that the trajectory (3.11) of the underdamped de Broglie–Bohm oscillator is identical to the formula for the *amplitude* decay of the oscillation predicted by classical mechanics for an underdamped oscillator.

On the other hand, the fact that the trajectory is independent of the mass may seem more surprising, but can also be understood physically if one recalls that, in classical mechanics, the trajectory involves the mass *exclusively* through the natural angular frequency of oscillation ω_0. (The amplitude decay of the oscillation is *independent* of the mass

and governed only by the friction constant γ if the damping force is written as above: $F_d = -2m\gamma\dot{x}$.) A non-oscillatory motion should not be expected to involve ω_0, and therefore the mass does not appear in (3.11). Once again, this is consistent with drawing the analogy between the de Broglie–Bohm trajectory and the amplitude decay of the oscillation of the classical underdamped trajectory.

4. Conclusion

We used here, as a 'guiding wave' in the de Broglie–Bohm hidden-variable theory, the wavefunction (2.5)–(2.8) which describes, according to some quantization schemes [17–23], an underdamped harmonic oscillator. It was possible to solve *without approximation* the de Broglie–Bohm equation of motion with this guiding wave in 'state' n, and the corresponding solution (3.11) was interpreted as the trajectory of the underdamped harmonic oscillator according to the de Broglie–Bohm theory. The physical validity of this interpretation rests, ultimately, on the interpretation of the quantum-mechanical wavefunction (2.5)–(2.8).

The de Broglie–Bohm trajectory possesses the properties that it is independent of the mass of the system and tends monotonically to the origin $x = 0$ as the time increases, features which contrast with the behaviour of the classical trajectory but were traced to a double origin: mathematically, the concept of 'quantum potential' enabled us to prove the equivalence between the underdamped de Broglie–Bohm oscillator and a critically damped classical oscillator, from which the properties of the de Broglie–Bohm trajectory can be deduced at once. From the physical point of view, some aspects of the motion were seen to follow from the fact that we guided the oscillator by a pure 'state' Ψ_n of the quantum-mechanical wavefunction, as opposed to a linear superposition of different 'states'.

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References

[1] de Broglie L 1926 *Compt. Rend.* **183** 447
[2] de Broglie L 1927 *Compt. Rend.* **184** 273; 1927 *Compt. Rend.* **185** 380
[3] Bohm D 1952 *Phys. Rev.* **85** 166, 180
[4] Bohm D, Hiley B and Kaloyerou P 1987 *Phys. Lett.* **144C** 321
[5] Bell J S 1989 *Speakable and Unspeakable in Quantum Mechanics* (Cambridge: Cambridge University Press) ch 4, p 33
[6] Nelson E 1966 *Phys. Rev.* **150** 1079
[7] Philippidis C, Dewdney C and Hiley B 1979 *Nuovo Cimento* B **52** 15

[8] Philippidis C, Bohm D and Kaye R 1982 *Nuovo Cimento* B **71** 75
[9] Bohm D 1953 *Phys. Rev.* **89** 458
[10] Bohm D and Vigier J-P 1954 *Phys. Rev.* **96** 208
[11] Bohm D and Hiley B 1984 *Found. Phys.* **14** 255
[12] Dewdney C and Hiley B 1982 *Found. Phys.* **12** 27
[13] Hirschfelder J and Christoph A 1974 *J. Chem. Phys.* **61** 5435
[14] Gonzalo Ares-de-Parga A, Rosales M and Jiménez J 1989 *Am. J. Phys.* **57** 941
[15] Kobe D, Reali G and Sieniutycz S 1986 *Am. J. Phys.* **54** 997
[16] Greenberger D 1979 *J. Math. Phys.* **20** 762
[17] Greenberger D 1979 *J. Math. Phys.* **20** 771
[18] Caldirola P 1941 *Nuovo Cimento* **18** 393
[19] Kanai E 1948 *Prog. Theor. Phys.* **3** 440
[20] Stevens K 1958 *Proc. Phys. Soc. Lond.* **72** 1027
[21] Kerner E 1958 *Can. J. Phys.* **36** 371
[22] Bopp F 1962 *Z. Angew. Phys.* **14** 699
[23] Svin'in I 1975 *Theor. Mat. Fiz.* **22** 107
[24] Weber J 1953 *Phys. Rev.* **90** 977
[25] Senitzky I 1960 *Phys. Rev.* **119** 670
[26] Edwards I 1979 *Am. J. Phys.* **47** 153
[27] Kennedy F and Kerner E 1965 *Am. J. Phys.* **33** 463
[28] Hasse R 1975 *J. Math. Phys.* **16** 2005
[29] Messer J 1979 *Acta Phys. Austriaca* **50** 75
[30] Helmholtz H 1887 *J. Reine Angew. Math.* **100** 137
[31] Mayer A 1896 *Ber. Ges. Wiss. Leipzig, Math.-Phys. Kl.* **49** 519
[32] Havas P 1957 *Nuovo Cimento Suppl.* **5** 363
[33] Bateman H 1931 *Phys. Rev.* **38** 815
[34] von Neumann J 1955 *Mathematical Foundations of Quantum Mechanics* (Princeton: Princeton University Press) ch 1, p 14, footnote 19
[35] Havas P 1956 *Bull. Am. Phys. Soc.* **1** 337
[36] Messer J 1978 *Lett. Math. Phys.* **2** 281
[37] Liboff R 1980 *Introductory Quantum Mechanics* (Reading, MA: Holden-Day) ch 7, p 175–9
[38] Chihong C 1993 *Mod. Phys. Lett.* A **8** 2037
[39] Isar A *et al* 1993 *J. Math. Phys.* **34** 3887